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FAST TRACK COMMUNICATION

The nonlinear fragmentation equation

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Abstract
We study the kinetics of nonlinear irreversible fragmentation. Here, fragmentation is induced by interactions/collisions between pairs of particles and modelled by general classes of interaction kernels, for several types of breakage models. We construct initial value and scaling solutions of the fragmentation equations, and apply the 'non-vanishing mass flux' criterion for the occurrence of shattering transitions. These properties enable us to determine the phase diagram for the occurrence of shattering states and of scaling states in the phase space of model parameters.

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Fragmentation is a phenomenon of the breaking up of particles into a range of smaller sized particles. It is naturally found in a wide variety of physical systems, ranging from comminution, breakup of grains, bubbles, droplets, polymer degradation, disintegration of atomic nuclei, etc. Fragmentation may occur through external forces, spontaneously, or through interactions/collisions between particles. The subject has been widely studied [1–10].

We are mainly interested in collision-induced nonlinear fragmentation as caused by binary interactions. Such systems can be described by the time evolution of $c(x, t)$, which is the number of particles of mass or size x at a given time t, or alternatively by its moments $M_n(t) = \int_0^\infty dx\, x^n c(x, t)$. Quantities with similar properties appear in coagulation processes. In either case the total mass is conserved, $M(t) = M_1(t) = 1$, while the total number of particles, $N(t) = M_0(t) = \int dx\, c(x, t)$, is not. In irreversible coagulation, the mean particle mass, $s(t) = M/N(t)$, increases monotonically, and may lead to a finite time singularity at $t = t_c$, the gelation transition, characterized by the appearance of an infinite cluster containing a finite fraction of the total mass, $\Delta(t) = 1 - M(t)$ (order parameter), where $\Delta(t) \neq 0$ for $t > t_c$. Alternatively, the gelation transition is characterized by a non-vanishing mass flux $\dot{\Delta}(t) = -\dot{M}(t)$ from finite size particles (sol) to the infinite cluster (gel) [11–13], i.e. a violation of mass conservation.

In irreversible fragmentation, the reversed scenario occurs. Here $s(t)$ is monotonically decreasing, while the overall mass is conserved. In these systems a finite time singularity may occur at t_c, the *shattering* transition. It is characterized by a non-vanishing mass flux, $\dot{\Delta}(t)$, i.e. the rate at which massive particles are converted into massless infinitesimals or fractal dust [1–3, 7]. If $\dot{\Delta}(t_c)$ is *finite* this transition has the character of a continuous phase transition, described by the order parameter $\Delta(t) = 1 - M(t)$, as in gelation [13]. In the case $\dot{\Delta}(t_c) = \infty$, *all* mass instantly 'evaporates' from the system; the transition is called explosive and also referred to as a first-order transition [8].

Smoluchowski's coagulation–fragmentation equation [11] gives the basic mean field description for reversible and irreversible coagulation [2, 3, 13] and fragmentation processes [1–10] in terms of the time evolution of $c(x, t)$ in spatially uniform (well-stirred) systems. In irreversible fragmentation or coagulation, the system is described by a nonlinear coagulation rate in combination with a spontaneous linear fragmentation rate and/or a collision- or reaction-induced nonlinear fragmentation rate. The system does not reach a steady state, but at asymptotically large times the distribution function $c(x, t)$ approaches under rather general conditions to the standard scaling form, which describes the typical x-dependence around the mean particle size $s(t)$, which is steadily decreasing.

The occurrence of shattering has been addressed only partially in the case of collision-induced nonlinear fragmentation. It shows a behaviour, qualitatively different from spontaneous (linear) fragmentation. Furthermore, the special cases analysed so far are not necessarily generic, but appear to be borderline cases. In this communication, we study the occurrence of shattering for general classes of fragmentation models within the framework of the nonlinear fragmentation equation and we analyse its peculiarities and point out the parallels with gelation.

Collision-induced irreversible fragmentation can be described at the mean field level by the nonlinear fragmentation equation with a collision term I composed of a loss and a gain term [4],

$$\partial c(x,t)/\partial t = I(x|c) \equiv -c(x)\int_0^\infty \mathrm{d}y\, K(x,y)c(y) + \int_x^\infty \mathrm{d}y \int_0^\infty \mathrm{d}z\, b(x|y)K(y,z)c(y)c(z). \tag{1}$$

Here $b(x|y)$ is a *conditional* probability, describing the distribution of outgoing fragments of mass x, given that a particle of mass y breaks [1–4]. One distinguishes: (i) *deterministic* or *splitting* models [4, 8], where a particle breaks into two equal fragments, hence $b(x|y) = 2\delta(x - y/2)$, and (ii) *stochastic* models, where a fragment of random mass x breaks off from a particle of mass y. As mass is conserved in a single breakup event, the outgoing fragment distribution has to obey the homogeneity requirement, $b(x|y) = y^{-1}b\big(\frac{x}{y}\big)$. For simplicity, we take the standard form $b(s) = (\beta+2)s^\beta$,[3] obeying

$$\int_0^y \mathrm{d}x\, x b(x|y) = y, \qquad \bar{N} = \int_0^y \mathrm{d}x\, b(x|y) = \frac{\beta+2}{\beta+1}. \tag{2}$$

For physical reasons the mean number of outgoing fragments satisfies $\bar{N} \geqslant 2$ which implies $-1 < \beta \leqslant 0$. Binary breakup corresponds to $\beta = 0$. In equation (1) we consider binary interactions, where the kernel $K(x, y)$ describes the interaction rate of pairs of particles (x, y). It may further contain a factor $\bar{p}(x, y)$, which gives the probability that breakage indeed occurs and may depend on the masses (x, y). If $\bar{p}$ is constant, it can be absorbed in the

[3] Choices for $b(x|y)$ have been discussed by Peterson [9] and Cheng and Redner [4] and choices for $a(x)$ by McGrady and Ziff in [2].

time scale. Of importance in our analysis is also the rate equation for the cumulative mass, $\dot{M}(x,t) \equiv \int_x^\infty \mathrm{d}y\, y\dot{c}(y,t)$, as can be derived from equation (1). It reads

$$\dot{M}(x,t) = -\int_x^\infty \mathrm{d}y \int_0^\infty \mathrm{d}z\, yB\left(\frac{x}{y}\right)K(y,z)c(y)c(z) \tag{3}$$

with $B(s) = \int_0^s \mathrm{d}u\, ub(u) = s^{\beta+2}$. In applications of the nonlinear Smoluchowski equation, a variety of collision kernels K has been proposed for processes induced by interacting particles [11, 13]. Because of mathematical simplicity, kernels of the *sum-product* form $K(x,y) = 1, x^p + y^p, (xy)^p, x^p y^q + x^q y^p$, etc have been extensively studied in coagulation processes. Physically motivated kernels are e.g. $K = (x^{-\alpha}+y^{-\alpha})(x^\beta+y^\beta)$ with $\alpha = \beta = 1/d$ for interaction rates among diffusing particles, or $K \sim (R_x + R_y)^{d-1}$ for ballistic collision rates in a d-dimensional system. Here $R_x \sim x^{1/d}$ is the radius of a particle of mass x and K is a geometrical cross-section. In most cases of physical interest, especially at limiting particle masses ($x \ll s(t)$ or $x \gg s(t)$), the kernels are continuous and homogeneous, i.e. $K(ax,ay) = a^\lambda K(x,y) = a^\lambda K(x,y)$ and $b(ax|ay) = a^{\lambda'} b(x|y)$ with $\lambda' = -1$. Kernels for coagulation can be classified by two exponents [13], i.e. $K(x,y) \sim x^p y^q$ if $x \ll y$, where $p \leqslant q$ and $\lambda = p + q$, with $p > 0$ (class I), $p = 0$ (class II) and $p < 0$ (class III), with the physical restrictions $\lambda \leqslant 2,\ q \leqslant 1$ [13]. This (p,q) classification appears to be relevant for nonlinear fragmentation as well, as we will show.

The breakage probability, $\bar{p}(x,y)$, defines three different types of models depending on whether particle x or y breaks: (i) *symmetric breakage*, where a randomly chosen particle of the interacting pair (x,y) breaks [4, 8] and where $\bar{p}(x,y) = 1$; (ii) *L-breakage*, where the larger particle breaks, hence $\bar{p}(x,y) = \theta(x-y)$; (iii) *S-breakage*, where the smaller particle breaks and $\bar{p}(x,y) = \theta(y-x)$; $\theta(x)$ stands for the unit step function. The corresponding nonlinear fragmentation equation for L-breakage is obtained from equation (1) by replacing $I(x|c)$ with $I_L(x|c)$, with $K(x,y)$ replaced by $K_L(x,y) = K(x,y)\theta(x-y)$ and similarly for S-breakage. Subsequently, we will discuss the nonlinear stochastic fragmentation equation for kernels of classes I, II and III for symmetric, L- and S-breakage models. Regarding exact solutions of the nonlinear fragmentation equation very little is known, and mostly restricted to mono-disperse initial conditions. The essential references are [4, 8], where the former contains a representative list of the older literature. Cheng and Redner [4] analyse the deterministic L- and S-breakage models, $K_L(x,y) = x^p\theta(x-y)$ and $K_L(x,y) = x^p\theta(y-x)$, and Krapivsky and Ben-Naim [8] does so for both the deterministic and the stochastic breakage models with $K_L(x,y) = \theta(x-y)$ and $K_L(x,y) = \theta(y-x)$.

Regarding the structure of the nonlinear integral-differential equations (1) and (3) for cases where $K(x,y) = a(x)a(y)$ is a general product kernel, it has been observed [4] that the nonlinear fragmentation equation can be transformed into a linear one with a new time variable $\tau(t)$ that is related to the physical time t in a nonlinear manner. The functional form of $\tau(t)$ determines whether a shattering transition is present or absent. So, to explain this dependence it is paramount to discuss how the initial solutions $c(x,t)$ of the nonlinear fragmentation equation for a given $c(x,0)$ can be constructed from the initial solutions $\bar{c}(x,\tau)$ of the linear fragmentation equation. To this end, we analyse the *linear* fragmentation

$$\partial c(x,t)/\partial t = -a(x)c(x) + \int_x^\infty \mathrm{d}y\, b(x|y)a(y)c(y), \tag{4}$$

where $a(x)c(x)$ represents the spontaneous or externally induced linear breakup rate. Exact solutions $c(x,t)$ are known for algebraic fragmentation rates, $a(x) = x^\alpha$ for all real α, and mono-disperse initial conditions, $c(x,0) = \delta(x-x_0)$ [1–3, 5]. These $c(x,t)$'s are the causal Green functions of equation (4) with a monomer source $\delta(t)\delta(x-x_0)$ [5]. So, $c(x,t) = 0$ for all $x > x_0$ at $t > 0$. In the following, we set $x_0 = 1$. Spontaneously fragmenting systems[4] with $\alpha \geqslant 0$ are non-shattering, i.e. the total mass $M(t) = 1$ at all times, and the total number of particles, $M_0(t) < \infty$ for all $t < \infty$. Moreover, moments with $n+\beta+1 > 0$ exist, and evolve for large t as $M_n(t) \sim t^{(1-n)/\alpha}$, while those with $n+\beta+1 \leqslant 0$ are divergent. On the other hand, spontaneously fragmenting systems with $\alpha < 0$ are shattering, and mass loss starts at the initial time. So, shattering occurs at $t = t_c = 0$, where $\dot{\Delta}(t_c)$ is finite; hence the transition is continuous. The possibility of an explosive shattering transition with $\dot{\Delta}(t_c) = \infty$ is never realized. All initial solutions, which have by definition $t > 0$, are non-mass conserving post-shattering solutions with $t > t_c = 0$. They behave for small x as $c(x,t) \sim A(t)x^{-\theta}$ with $\theta = \alpha + 2$. Consequently, moments $M_n(t)$ with $n \leqslant 1+\alpha$ are divergent for *all* times and those with $n > 1+\alpha$ decay for long times as $M_n(t) \propto A(t) \propto \mathrm{e}^{-t}t^{(\beta+2)/\alpha}$, including $M(t)$. We also point out that for $\alpha > 0$ the exact solutions converge asymptotically to a *standard* scaling form, $c(x,t) = (1/s^2(t))\varphi(x/s(t))$ [1–9], where the scaling limit is formally defined as the coupled limit, $t \to \infty$ and $x \to 0$ with $x/s(t)$ kept constant. In this limit, where $s \sim 1/M_0 \sim t^{-1/\alpha} \to 0$ (i.e. the total number of particles $M_0(t)$ diverges), the exact solution becomes $s^2 c(su,t) \sim \varphi(u) \sim u^\beta \exp[-u^\alpha]$. Those with $\alpha < 0$ do not approach a scaling form. Inspection of the exact solution as $\alpha \to 0$ at fixed (x,t) shows that $c(x,t) = 0$ for all $x > x_0 = 1$ and reads for $x < 1$ (see [2]),

$$c(x,t) = \mathrm{e}^{-t}(2t/\ln(1/x))^{1/2} I_1[2(2t\ln(1/x))^{1/2}], \tag{5}$$

where $s(t) = \mathrm{e}^{-t}$ and $I_1(x)$ is the modified Bessel function of integer order $n = 1$. This expression shows that the borderline case, $\alpha = 0$, is exceptional, i.e. non-scaling *and* non-shattering.

Let us now consider the *nonlinear* fragmentation equation for symmetric breakage with product kernel $K(x,y) = a(x)a(y) = (xy)^p$ and $0 \leqslant \lambda = 2p \leqslant 2$. In this case, equation (1) is a quasi-linear equation for which exact initial value solutions can be obtained. It can be mapped onto the linear fragmentation equation with $\alpha \to p$ and $t \to \tau$, defined through $\mathrm{d}\tau = M_p(t)\,\mathrm{d}t$. Consequently the mass distribution, $\bar{c}(x,\tau)$, and moments, $\bar{M}_n(\tau) = \int_0^1 \mathrm{d}x\, x^n \bar{c}(x,\tau)$, for mono-disperse initial values are known explicitly, and only $\tau(t)$ needs to be determined in order to have the complete solution as a function of t. For $p = \alpha > 0$, where K is a class I kernel, total mass is conserved for all τ, and the moments for $n \neq 1$ behave at large τ as $\bar{M}_n(\tau) \sim \tau^{(1-n)/p}$. Furthermore, to have τ as a function of t we need to invert the relation $t = \int_0^\tau \mathrm{d}s/\bar{M}_p(s)$. If $\lambda = 2p > 1$, then $t(\tau) \sim \tau^{2-1/p}$ is monotonically increasing, the relation is invertible and $t \to \infty$ as $\tau \to \infty$. Hence, $M_1(t) = 1$ for all t, and there is no shattering and no divergence of $M_0(t)$ at any finite time. However, if $\lambda = 2p < 1$, then as $\tau \to \infty$, $t \to t_c \equiv \int_0^\infty \mathrm{d}\tau/M_p(\tau) < \infty$. Consequently there exists a finite time singularity, $\tau \sim (t_c - t)^{-p/(1-2p)}$ as $t \to t_c$ and mass remains conserved only for $t < t_c$ and vanishes instantaneously at t_c, where $\dot{\Delta}(t_c) = -\infty$. At the same time all moments, behaving as $M_n(t) \sim (t_c - t)^{(1-n)/(1-2p)}$, either diverge or vanish. These are the hallmarks of an explosive shattering transition at t_c, where all massive particles are converted instantaneously into fractal dust. For $\lambda = 2p < 0$ (class III), the kernel $K = (xy)^p$ can be mapped on the linear equation through $\mathrm{d}\tau = M_{-|p|}(t)\,\mathrm{d}t$. Its moments $\bar{M}_n(\tau)$ with $n < 1+\alpha = 1-|p|$ do not exist. Consequently $\tau(t)$ is not defined, and $c(x,t)$ does not exist for mono-disperse initial conditions with class III kernels. The same applies to scaling solutions. The corresponding class II kernel with $p = \alpha = 0$ or $K = 1$ represents an exceptional point, as discussed below equation (5). The solution at $p = 0$ ($K = 1$, class II) exists, is shattering and $\bar{c}(x,\tau)$ is identical to the non-generic, non-scaling solution of the linear equation (4) at $\alpha = 0$.

[4] The analytic results in this paragraph have been calculated with the help of [7].

To determine possible *scaling* solutions, we substitute the scaling ansatz $c(x,t) = (1/s^2)\varphi(u = x/s)$ in (3) and take the derivative, yielding the scaling equation for symmetric, L- and S-breakage ($A = 0, L, S$),

$$\left(\varphi(u)/u^{\beta}\right)' = -(\varphi(u)/\gamma u^{\beta+1}) \int_0^{\infty} \mathrm{d}v\, K_A(u,v)\varphi(v), \tag{6}$$

where γ is an arbitrary positive separation constant, $\varphi(u)$ has to satisfy the boundary condition, $u^2\varphi(u) = 0$ as $u \to \infty$, and $K_0 = K$. For all three types of breakage models the evolution equation for the mean particle size is the same, $\dot{s}s^{-\lambda} = -\gamma$. Its solution is

$$s(t) \sim \begin{cases} (t_0 + t)^{1/(1-\lambda)} & \text{for} \quad \lambda > 1, t \to \infty \\ \exp[-\gamma t] & \text{for} \quad \lambda = 1, t \to \infty \\ (t_c - t)^{1/(1-\lambda)} & \text{for} \quad \lambda < 1, t \leqslant t_c, \end{cases} \tag{7}$$

where $t_0, t_c \to \infty$ as $\lambda \to 0$. The appearance of the finite time singularity at t_c indicates that shattering only occurs for $\lambda < 1$. Systems with $\lambda \geqslant 1$ are non-shattering [4]. Note that the scaling limit in the pre-shattering critical region is defined as the coupled limit: $t \uparrow t_c$ and $x \to 0$ with $x/s(t) = \text{constant}$.

For sum-product kernels in symmetric breakage the rhs in equation (6) reduces to sums of powers u^s, multiplied by coefficients $m_n = \int_0^{\infty} \mathrm{d}u\, u^n \varphi(u)$, which can be determined self-consistently. Specifically, for $K(x,y) = (xy)^p$ ($p > 0$, class I), one obtains from the rhs of (6) $\psi'(u) = \psi(u) m_p u^{p-1}/\gamma$, where $\psi(u) \equiv \varphi(u)/u^{\beta}$. This can be readily integrated to yield

$$\varphi(u) = C u^{\beta} \exp[-u^p m_p/\gamma p], \tag{8}$$

where C and γ are determined by imposing normalization ($m_0 = 1$) and mass conservation ($m_1 = 1$). In order to get simpler analytic expressions, we use the invariance property that the scaled distribution, $\bar{\varphi}(\bar{u})$, obtained under the similarity transformation $\bar{\varphi}(\bar{u}) = s_0^{-2}\varphi(u/s_0)$ for an arbitrary constant s_0 also satisfies equation (6) with $\bar{m}_1 = 1$. This property allows us to fix γ by setting $m_p/\gamma p = 1$, which is a self-consistency equation. With this choice the moments read $m_n = (C/p)\Gamma(b_n) = \Gamma(b_n)/\Gamma(b_1)$, where $\Gamma(x)$ is the gamma function, and $b_n = (1 + \beta + n)/p$. Hence, the scaling distribution function can be expressed as $\varphi(u) = p u^{\beta}\, \mathrm{e}^{-u^p}/\Gamma((\beta+2)/p)$. Solutions with a different normalization, e.g. $m_0 = 1$, are easily derived using the invariance property under similarity transformations.

Similarly one derives that the size distribution for sum kernels, $K = x^p + y^p$ (class II) with $p > 0$, has the form $\varphi(u) = C u^{\bar{\beta}}\, \mathrm{e}^{-u^p} = p u^{(\beta-1)/2}\, \mathrm{e}^{-u^p} / \Gamma((\beta+3)/2p)$ with $\bar{\beta} = \beta - m_p/\gamma$. Here we impose the self-consistent equation $m_0/p\gamma = 1$ to determine the separation constant γ. Imposing mass conservation leads to the second equality in the previous equation. The moments of the distribution can then be computed; in particular $m_p = b_0 \gamma p$ where now $b_n = (1 + \bar{\beta} + n)/p$ which implies $\bar{\beta} = \frac{1}{2}(\beta - 1)$. The exact scaling solutions for the symmetric breakage kernels (p, p) and $(0, p)$ above have different limiting forms as $p \to 0$. So, the analysis starting below equation (5) shows that the K kernel with $(p, q) = (0, 0)$ is quite singular. In a similar manner the scaling solutions $\varphi(u)$ for the geometric collision cross-section, $K \sim (x^{1/3} + y^{1/3})^2$, and closely related kernels can also be found, as well as the asymptotics of $\varphi(u)$ for general class I and II kernels in all breakage models of type $A = (0, L, S)$. In L- and S-breakage models for *generic* K no exact initial value or scaling solutions are known, except for the non-generic borderline case $K = 1$ in [8], which lacks standard scaling in the variable $u = x/s(t)$ in all three breakage models.

To analyse from a broader perspective the occurrence of shattering, we will focus on the behaviour of the cumulative mass flux for vanishingly small masses. If $\lim_{x\to 0} \dot{M}(x,t) \equiv \dot{M}(t)$ is *vanishing* at t_c, the system is non-shattering; otherwise there is shattering. If $-\infty <$

$\dot{M}(t_c) < 0$, the phase transition is *continuous*, and $c(x, t)$ exists for $t > t_c$. If $\dot{M}(t_c) = -\infty$, the phase transition is explosive (first order), and $c(x, t)$ does not exist for $t > t_c$.

For general $K(x, y)$ in classes I, II and III, the mass flux $\dot{M}(x, t)$ can only have a non-vanishing limit for $x \to 0$ if $c(x, t)$ is of the power-law type because the rhs contains the factor $x^{\beta+2}$ with $-1 < \beta \leqslant 0$. So we propose the post-shattering ansatz $c(x, t) \sim A(t)x^{-\theta}$ $(x \to 0)$ and determine θ such that $\dot{M}(t) \neq 0$ (see [12, 13]). Moreover, $\theta < 2$ because the total mass should remain finite.

The evolution equation for fragmentation with S-breakage is described by equation (3) with K replaced by K_S. Inserting the ansatz above yields for small x,

$$\dot{M}(x, t) \simeq -A^2 k(\theta)(x^{3+\lambda-2\theta})/(2\theta + \beta - 1 - \lambda)$$
$$k(\theta) = \int_1^\infty \mathrm{d}s\, K(1, s) s^{-\theta} \qquad (1 + p < \theta < 2), \tag{9}$$

where $K(1, s) \sim s^p$ for $s \gg 1$. In the case $\theta = \frac{1}{2}(3+\lambda)$ the above small-x limit yields a finite result for the mass flux $\dot{M}(t) = -A^2(t)k(\theta)/(2+\beta)$, i.e. it allows the existence of a *continuous* shattering transition with a post-shattering solution of the algebraic form for $t > t_c$; $\theta < 2$ implies $\lambda < 1$. At the (unknown) shattering time t_c mass conservation breaks down, and for $t > t_c$ there exists a non-vanishing order parameter $\Delta(t) = 1 - M(t) > 0$ with $\dot{\Delta}(t) \sim A^2(t)$. Equation (9) also includes the special result, obtained in [8] for the S-breakage model with $K = 1$ and $\beta = 0$.

It is remarkable that the post-gelation distribution, $c(x, t) \sim A(t)x^{-\theta}$, occurring in Smoluchowski's coagulation equation for $\lambda > 1$, has the same exponent $\theta = (3 + \lambda)/2$ [12, 13] as in the fragmentation process above, hence the close analogy between gelation and continuous shattering. Note that the value of exponent β has no influence on the existence of shattering.

A similar analysis can be performed for symmetric and L-breakage models. In doing so, we introduce a lower cut-off ϵy on the z-integral in equation (3) and take $\lim_{\epsilon\to 0}$ at the end of the calculations. Due to the physical restrictions on the allowed values for p and q, shattering is always explosive rather than continuous.

From the properties discussed in this communication, we can construct the phase diagram for symmetric breakage in the (p, q)-plane. It is restricted to the triangular region, spanned by $(0, 0)$, $(0, 1)$, $(1, 0)$ and includes the boundaries. The region with $0 \leqslant \lambda < 1$ represents shattering systems, and the region with $1 \leqslant \lambda \leqslant 2$ represents non-shattering ones. The whole triangular region shows standard scaling in the variable $u = x/s(t)$, except in the singular corner $(0, 0)$. Regarding the phase diagram for L- or S-breakage the location of the left boundary (separatrix between 'non-existence' and 'existence of scaling solutions'), including the singular point $(0, 0)$, is unknown, and the behaviour on it may be different from its right and left limits. From [8] it is known that a new type of scaling in the variable $x/m^*(t)$ appears at $(0, 0)$. Here m^* is a characteristic mass that cannot be defined *a priori*, but follows from a clever mapping of the fragmentation equation on the nonlinear equation for travelling fronts. In contrast to models with symmetric breakage, which are quasi-linear, the scaling equations for L- and S-breakage are genuinely nonlinear, i.e. $\varphi''(u) = F(u, \varphi', \varphi)$, and the only solutions known are those for the singular point $(0, 0)$.

We have discussed the generic behaviour of collision-induced irreversible fragmenting systems at the mean-field level. We have shown that the scenarios for nonlinear fragmentation are qualitatively different from those of the spontaneous linear fragmentation. The behaviour of the shattering transition depends both on the kind of fragmentation kernel and on the type of breakage. For symmetric and L-breakage, where the kernel K has a degree of homogeneity $\lambda < 1$, shattering is always explosive, while S-breakage models show a continuous shattering transition, analogous to gelation. The existence of a transition does not depend on the details of the fragment distribution, $b(x|y)$, i.e. on β. Shattering in collision-induced fragmentation always takes place at a finite time $t_c \neq 0$, as opposed to linear fragmentation where shattering occurs at $t_c = 0$ for $\alpha < 0$. In contrast to gelation [13], in class III kernels with symmetric breakage, neither initial nor scaling solutions exist. The solutions for fragmentation models with a fragment distribution, $b(s) = 0$ for $s < s_0, b(s) \neq 0$ for $s_0 < s < 1$, have scaling solutions $\varphi(u)$ exhibiting log-normal distributions at small u [4].

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